Research statement

Alexey Okunev
abo5297@psu.edu

1 Overview

My research interest is in Dynamical Systems and their applications. I am currently on my second postdoc at Pennsylvania State University. My first postdoc was at Loughborough University (UK), where I worked with Anatoly Neishtadt on perturbations of integrable systems. My PhD research (under supervision of Yulij Ilyashenko, at Higher School of Economics, Moscow, Russia) was on discrete dynamical systems; I worked on problems involving iterated function systems, attractors, hyperbolicity and partial hyperbolicity.

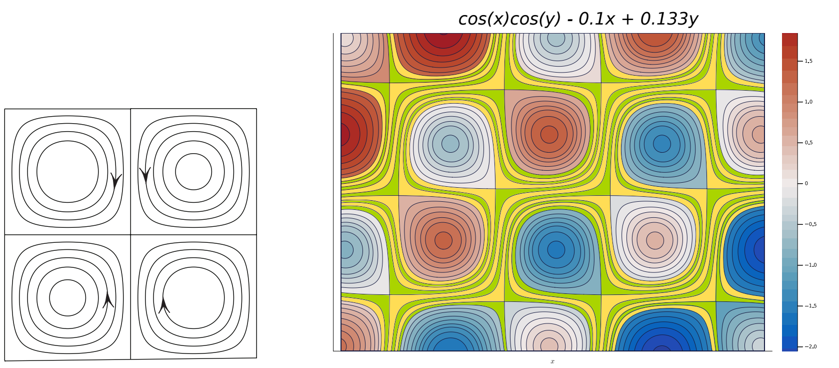
2 Thick Arnold tongues

My most recent completed project ([1], joint with Mark Levi) is a study of a physically motivated dynamical system exhibiting nontrivial and perhaps unexpected properties. The mathematical explanation involves perturbation theory and circle maps dynamics, and this system also has connections with number theory and topology. The example we study originates from particle transport in fluid flows: it is a simplified mathematical model of a small inertial particle drifting in a fluid and pulled by gravity proposed by Maxey and Corrsin in 1986, [2]. The fluid motion in this example is given by the *cellular flow*, a periodic Hamiltonian flow on the plane along the level curves of $H = \cos x \cos y$ (Figure 1a). We are interested in the case when the cells are not aligned with gravity. As the vertical direction is already booked by cell orientation, the direction of the gravity becomes a parameter given by a vector $(b, a) \in \mathbb{R}^2$. We call this vector *forcing*, and it can be interpreted as terminal fall velocity of the particle in still fluid.

When the particle is so small that the effect of its inertia is negligible (*zero inertia case*), its movement is given by the following Hamiltonian vector field (Figure 1b):

$$\mathbf{v}(x, y) = \Big(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\Big), \qquad H(x, y) = \cos x \cos y - ax + by. \tag{1}$$

This model (1) was introduced in the context of particle transport as early as 1949 by Stommel [3].



(a) Cellular flow.

(b) Hamiltonian flow $\mathbf{v}$. Regions with unbounded trajectories are highlighted in green and yellow.

Figure 1: Fluid flow and Hamiltonian flow $\mathbf{v}(\mathbf{x})$ describing particle movement for zero inertia case.

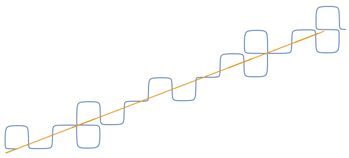
A particle with small but nonzero inertia has a velocity of its own; however, the fluid drag force proportional to the velocity mismatch between the particle and surrounding fluid makes its motion close to the motion of a zero inertia particle.[1] Let $\mathbf{x} = (x, y)$ and $\dot{\mathbf{x}} = (\dot{x}, \dot{y})$ denote the particle's position and velocity and let $\varepsilon > 0$ be a small parameter describing the inertia strength. Then particle's movement can be modeled by a second order ordinary differential equation:

$$\ddot{\mathbf{x}} = -\frac{1}{\varepsilon}(\dot{\mathbf{x}} - \mathbf{v}(\mathbf{x})), \qquad \mathbf{x} \in \mathbb{R}^2. \tag{2}$$

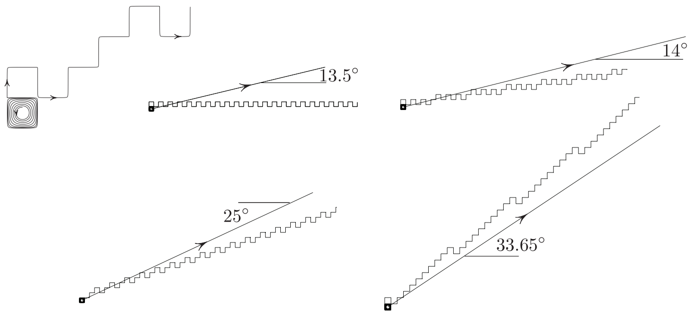
[1] A simple computation shows that the total force acting on the particle (the fluid drag plus gravity) is proportional to the difference between the actual particle velocity and the velocity $\mathbf{v}$ a zero-inertia particle would have.

Such a model is a simplified case of the Maxey-Riley model [4] and is often used in the literature on particle transport; cellular flow is also a very common choice of the fluid flow. While a spherical particle moving in a cellular flow without the gravity would simply approach the cell boundaries, many variations of this setup exhibit nontrivial dynamics. To name just a few references (more can be found therein), there are studies of particles in a cellular flow that are influenced by random thermal noise [5], are asymmetric [6] or compound [7], as well as studies of particles in time-dependent cellular flow [8].

To describe the particle motion given by (2) and (1), consider the trajectories of the Hamiltonian vector field $\mathbf{v}$ first (Figure 1b). Those are simply level curves of $H(x, y)$. Assuming that the forcing (b, a) is fairly small, most of the trajectories are bounded, but some "channels" formed by unbounded trajectories appear. All unbounded trajectories are at a finite distance from a straight line with the slope a/b, as $H = -ax + by + O(1)$ is constant on the trajectories. As a side note, the combinatorics of these unbounded trajectories is quite intriguing (Figure 2a) and is connected to symbolic dynamics as well as to number-theoretical properties of b/a. Additionally, the picture in Figure 1b is an (easy) special case of a beautiful problem by Sergei Novikov ([9], see also surveys [10] and [11]): to study the topology of a foliation of a closed surface (in our case the torus, as $\mathbf{v}$ is periodic) given by a closed 1-form on it (in our case, dH).



(a) Unbounded trajectory of the Hamiltonian flow $\mathbf{v}$ follows a line with the slope a/b.



(b) Particle drift for different forcing directions.

Figure 2: A trajectory of the Hamiltonian flow $\mathbf{v}(\mathbf{x})$ and particle drift according to (2).

Our interest in this problem was motivated by unusual properties of *inertial* particle motion, as modeled by (2) with $\mathbf{v}$ given by (1), that we first observed in numerical experiments. While inertial particle trajectories also have an asymptotic slope that does not depend on the initial data, the direction of the particle drift does not coincide with the forcing direction (Figure 2b). Moreover, as Figure 2b (top) illustrates, a small 0.5° change in the direction of the forcing causes a drastic change of the direction of the trajectory.

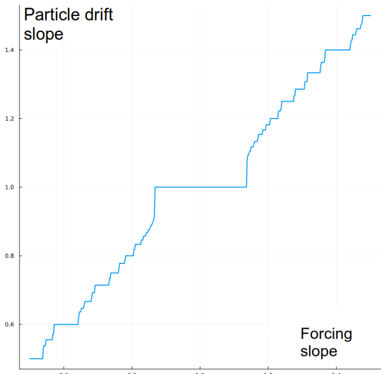
We **investigate how the direction of particle drift depends on the slope** a/b **of the external forcing** and prove that when a and b are positive but fairly small[2] for any small enough $\varepsilon > 0$

- Particle drift slope ρ exists and does not depend on the initial data $(\mathbf{x}(0), \dot{\mathbf{x}}(0))$, given that $(\mathbf{x}(0), \dot{\mathbf{x}}(0))$ is outside of a very small subset of $\mathbb{R}^4$ given by a countable union of codimension one manifolds.
- For fixed b and ε, the dependence of particle drift slope on the forcing slope $\alpha = \frac{a}{b}$ is a Cantor-like function with steps at all rational values of ρ (Figure 3a).
- The function $\rho(\alpha)$ is steep at the endpoints of the steps: if one takes α_0 at the right endpoint of any step, for any $n \in \mathbb{N}$ we have $\lim_{\Delta\alpha \to 0^+} \frac{(\rho(\alpha_0+\Delta\alpha)-\rho(\alpha_0))^n}{\Delta\alpha} = \infty$.[3] This explains the abrupt changes of ρ we observed in numerics.
- The set of the values of α such that $\rho(\alpha)$ is irrational (equivalently, the complement to the union of the steps) has zero Hausdorff dimension (and hence zero Lebesgue measure as well).

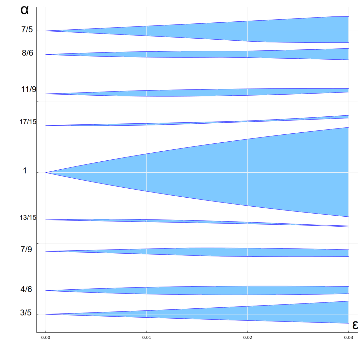
In the above description, the parameter ε in (2) was fixed. A fuller picture can be seen when ε is varied as well: for each rational number $\frac{p}{q}$, one can draw the set (called a *tongue*) of all pairs (ε, α) giving particle drift slope equal to $\frac{p}{q}$ (Figure 3b). The picture we get is similar to the famous Arnold tongues example [12], but, unlike it, the union of all tongues occupies a full Lebesgue measure – hence we use the name "thick Arnold tongues" for this example.

[2] We assume $a, b \in (\gamma, \delta)$ with δ small enough; then there exists $\varepsilon_1 = \varepsilon_1(\gamma, \delta)$ such that conclusions hold for all positive $\varepsilon < \varepsilon_1$.

[3] Which is like the inverse function to the classic Calculus example $y = e^{-1/x}$. It was a pleasant surprise to find such a function outside a textbook.



(a) $\rho(\alpha)$ for fixed ε



(b) Arnold tongues for $\rho(\varepsilon, \alpha)$. If all the tongues were shaded, blue color would occupy full measure.

Figure 3: Particle drift slope ρ as a function of forcing slope α and inertia parameter ε.

This description is obtained using tools from perturbation theory and circle maps dynamics. As observed by Rubin, Jones, and Maxey [13] using Fenichel's results [14] on normal hyperbolicity, the system (2) in $\mathbb{R}^4 \ni (\mathbf{x}, \dot{\mathbf{x}})$ admits a two-dimensional attracting invariant manifold and the restriction of the dynamics to it (using $\mathbf{x}$ as a coordinate) is a small (of order ε) dissipative perturbation of the Hamiltonian flow $\mathbf{v}(\mathbf{x})$ given by (1). As $\mathbf{v}(\mathbf{x})$ is periodic, the particle dynamics given by (2) is periodic in $\mathbf{x}$, and this manifold is a covering space of the 2-torus. This reduces (2) to a torus flow depending on the parameter $\alpha = a/b$ (we assume b and ε are fixed, then $a(\alpha) = \alpha b$). Taking the first return map further reduces the problem to the study of a family of circle maps possessing a special feature: they have two *flat spots*, that is, two intervals where these maps are constant. The particle drift slope coincides with the rotation number of the circle map up to a linear scaling.

Circle maps with flat spots were considered by many authors and appear naturally as first return maps for Cherry flows on torus and as truncations of non-invertible circle endomorphisms. We build on the works of Boyd [15] and Veerman [16] to understand how the rotation number depends on the parameter α in the case at hand. This involves a careful study of the properties of the circle map family originating from the model of particle transport and generalizing Veerman's theorem [16] claiming zero Hausdorff dimension of the set of parameters with irrational rotation number for families of circle maps with one flat spot to the case of an arbitrary number of flat spots.

3 Averaging and separatrix crossings

3.1 A brief non-technical overview

Separatrix crossings for small perturbations of integrable systems are important in many applications such as celestial mechanics and dynamics of charged particles. My favorite example is the work [17] by Jack Wisdom describing how consecutive separatrix crossings change the orbits of asteroids in a resonance with Jupiter resulting in the formation of the Kirkwood gaps within the main asteroid belt.[4] To introduce separatrix crossings for readers with interests far from perturbation theory, let us examine the following example. Consider a particle with position and velocity $x, \dot{x} \in \mathbb{R}$ in a double-well potential $V(x)$, Figure 4a. Then let us add to this *unperturbed system* a small dissipative force εf acting on it, called the *perturbation*:

$$\ddot{x} = -\frac{\partial V}{\partial x} + \varepsilon f(x, \dot{x}). \tag{3}$$

The unperturbed system is a completely integrable Hamiltonian system, and the energy $E = \frac{\dot{x}^2}{2} + V(x)$ is a first integral. To study the change of energy due to the perturbation, one can average the rate of change of E over periodic solutions of the unperturbed system and get a one-dimensional ODE. This system is called the *averaged system*, Figure 4b. If the energy for the averaged system decreases (e.g., f includes friction), then the particle will eventually be trapped into one of the two wells. This is called separatrix crossing, as the solution of (3) goes through a separatrix of the unperturbed system, the figure-eight shape in Figure 4a. Note that the particle can be captured into each of the two wells[5], so one has to consider two solutions of the averaged system. Then one of them will be close to the actual evolution of the energy, with the error $O(\varepsilon \ln \varepsilon)$, [18, 19].

[4] This mechanism is also considered to be a possible source of Earth-crossing asteroids.

[5] And order one averaging method cannot determine which well it will be captured in.

One also has to remove a small measure of initial data that "cannot decide" between the two wells and gets trapped near the point C between them.

(a) Unperturbed system. (b) Averaged system and resonant zones.

Figure 4: Example of separatrix crossing.

Suppose now that the perturbing force periodically depends on the time: $\ddot{x} = -\frac{\partial V}{\partial x} + \varepsilon f(x, \dot{x}, t)$ with $f(x, \dot{x}, t) = f(x, \dot{x}, t+1)$. Then, in order to obtain the averaged system, one averages with respect to both φ and t, where $\varphi \in [0, 2\pi]$ is the angle from the pair of action-angle variables. These variables parametrize periodic trajectories of unperturbed system in such a way that $\dot{\varphi}$ is constant along each trajectory. The unperturbed dynamics of (φ, t) for fixed E is a linear torus flow, and typically (for irrational winding) the time average will coincide with the spatial average. However, this is not the case if E corresponds to a rational winding of the torus, and this leads to a larger error of averaging. Such values of E are called *resonant*, and when E changes, it crosses resonant values, Figure 4b.[6] For most initial data, crossing a resonance causes a $O(\sqrt{\varepsilon})$ deviation from the averaged system, but a small measure of initial data can be captured into the resonance − that is, $E(t)$ gets trapped near the resonant value for a time of order $1/\varepsilon$, leading to order 1 deviation from the averaged system.

In [20] (joint with A. Neishtadt, submitted), we rigorously **estimate the accuracy of averaging method for small perturbations of completely integrable two-frequency Hamiltonian systems near separatrices**. One can keep in mind the example above, where the two angular variables are φ and t. However, our setting is perturbations of general two-frequency systems, and we also allow the Hamiltonian to depend on parameters slowly changing due to the perturbation. We show that for any initial data outside a small measure set corresponding to solutions close to capture into a resonance, the corresponding solution of the perturbed system is approximately described by the averaging method with the accuracy $O(\sqrt{\varepsilon})$. This accuracy cannot be improved, as a single passage through resonance causes such error. This result extends Neishtadt's Theorem (1975, [21]) on two-frequency systems far from separatrices to the general case with separatrix crossings and provides rigorous justification for the use of averaging method in this setting, which is frequently encountered in applications. This was a challenging problem with many difficulties to overcome. To name one, far from separatrices there are only finitely many resonances such that capture is possible, while in our setting there can be an infinite number of resonances allowing for capture, and these resonances accumulate on the separatrices.

A side result of the same project with A. Neishtadt is an estimate [22] on the accuracy of order two averaging method for non-Hamiltonian perturbations of one-frequency systems. For the example above, this is the case when f does not depend on t. Order 2 averaging allows one to track the evolution of the angle φ after separatrix crossing, which is impossible using the standard (order one) averaging method. In particular, for our example order 2 averaging describes in which well the particle will be captured. Such formulas for the angle φ after a separatrix crossing were obtained before for various settings when the perturbed system is also Hamiltonian [23, 24].

3.2 Background: perturbations of integrable systems and averaging method

The averaging method is a classical powerful tool in perturbation theory of dynamical systems. In particular, it is widely used for perturbations of completely integrable Hamiltonian systems. Consider a Hamiltonian system

$$\dot{p} = -\frac{\partial H}{\partial q}, \qquad \dot{q} = \frac{\partial H}{\partial p}, \qquad p, q \in \mathbb{R}^n,$$

where the Hamiltonian $H(p, q, z)$ may also depend on a parameter $z \in \mathbb{R}^k$. Assume that this system is completely integrable, i.e., there exist action-angle variables $I \in \mathbb{R}^n$, $\varphi \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ such that in these

[6] one can leave only $O(|\ln \varepsilon|)$ resonant values, as others have negligibly small effect on the dynamics

variables the dynamics writes in the form

$$\dot{I} = 0, \qquad \dot{z} = 0, \qquad \dot{\varphi} = \omega(I).$$

Completely integrable systems with $\omega \in \mathbb{R}^n$ are often called n-frequency systems. Action-angle variables do not necessarily exist globally, there may be several domains with their own action-angle variables, and the boundaries between such domains are called *separatrices*.

Now, consider a small perturbation of the integrable system:

$$\dot{p} = -\frac{\partial H}{\partial q} + \varepsilon f_p(p, q, z), \qquad \dot{q} = \frac{\partial H}{\partial p} + \varepsilon f_q(p, q, z), \qquad \dot{z} = f_z(p, q, z). \tag{4}$$

The actions I of the unperturbed system and the parameter z are the slow variables of the perturbed system, and change by $O(1)$ over time intervals of order at ε^{-1}. As the phases φ change much faster, solving such systems numerically is often unfeasible. Instead, the averaging method is usually used: one averages the rates of change of the slow variables over the phases φ, and writes the *averaged system*

$$\dot{\bar{I}} = \bar{f}_I(\bar{I}, \bar{z}), \qquad \dot{\bar{z}} = \bar{f}_z(\bar{I}, \bar{z}). \tag{5}$$

Then the evolution of slow variables along a solution of (4) can be approximated by a solution $(\bar{I}(t), \bar{z}(t))$ of the averaged equation.

The simplest setting is a perturbation of a one-frequency system far from separatrices, then averaging method works for all initial data with accuracy $O(\varepsilon)$ over time intervals of order ε^{-1}. Two main obstructions that arise in more complicated settings are resonances and separatrix crossings.

Separatrix crossings are possible even for one-frequency systems: due to the perturbation solutions of the perturbed system can cross separatrices of the unperturbed systems. As the angle φ from the pair of action-angle variables is undefined on separatrices, the equations used to justify the averaging method have a singularity there. Additionally, a small fraction of solutions spends a large time near the separatrix (e.g. in the example in Figure 4a the particle can be trapped at the point C between the two potential wells). In the one-frequency case, separatrix crossings are well studied; in particular, the change of slow variables along most[7] solutions of (4) (with one-dimensional p and q) is described by a solution of the averaged system with the accuracy $O(\varepsilon)$ before separatrix crossing and $O(\varepsilon|\ln\varepsilon|)$ after it ([25]).

Resonances appear starting with two-frequency systems, and for simplicity we will restrict ourselves to the case of two frequencies only. Then, $\omega(I) = (\omega_1(I), \omega_2(I))$, and a *resonance* happens when ω_1 and ω_2 are commensurate: $\frac{\omega_2}{\omega_1} = \frac{s_2}{s_1} \in \mathbb{Q}$. This is a codimension one manifold in the space of slow variables $\mathbb{R}^n_I \times \mathbb{R}^k_z$. At a resonance the trajectories of the fast system $\dot{\varphi}_1 = \omega_1$, $\dot{\varphi}_2 = \omega_2$ do not fill the whole torus, and averaging may thus fail near resonances. Indeed, it turns out that a small measure of initial data (of order $\sqrt{\varepsilon}$) can be *captured* in a resonance (Figure 5a, this means that the resonance is preserved along solutions of the perturbed system for a large time). Most solutions pass through the resonance, but experience *scattering*, a jump of the magnitude $O(\sqrt{\varepsilon})$, Figure 5b. Far from separatrices there are only finitely many resonances such that capture is possible (corresponding to rational numbers s_2/s_1 with small numerator and denominator).

(a) Capture into a resonance. (b) Scattering on a resonance.

Figure 5: Crossing a resonance.

Neishtadt's theorem ([26], see also [27]) justifies the use of averaging method for small perturbation of two-frequency integrable systems *far from separatrices*. It claims that under some genericity conditions for any initial data outside an exceptional set of measure $O(\sqrt{\varepsilon})$ the evolution of $I(t)$ and $z(t)$ along the solutions of perturbed system (3) is $O(\sqrt{\varepsilon})$ close to a solution of the averaged system (5) over times of order ε^{-1}.

[7]The exceptional set corresponds to solutions that are trapped near separatrices; it can be taken to have the measure $O(\varepsilon^r)$. for any $r > 0$

3.3 Separatrix crossings for perturbations of two-frequency systems

Theorem *([20], joint with A. Neishtadt). Consider a perturbation* (4) *of a two-frequency integrable system. Suppose certain nondegeneracy conditions*[8] *are satisfied. Then when* ε *is small enough, for any initial data except a set of small measure* $O(\sqrt{\varepsilon}|\ln \varepsilon|^5)$ *the accuracy of the averaging method is* $O(\sqrt{\varepsilon})$ *even for systems with separatrix crossings.*

This result is valid for the general case (i.e., the perturbed system can be non-Hamiltonian), but it is also new for the case when the perturbed system is Hamiltonian (e.g., for slow-fast Hamiltonian systems – then the slow variables can be treated as a parameter z).

Details. Compared with the setting of Neishtadt's theorem (i.e., far from separatrices), we had several conceptual difficulties, in addition to many technical ones. Resonant zones such that capture is possible can accumulate on separatrices (Figure 4b), while far from separatrices there are only finitely many such resonances. Action-angle variables have a singularity on the separatrices, and some of the functions describing averaging become unbounded, adding extra complexity to the study of the averaging method near separatrices. In particular, this also applies to resonances near separatrices. There is a standard procedure that describes the passage through a resonant zone using an auxiliary equation, an equation of pendulum type with a small perturbation. In our case, if this procedure is applied straightforwardly, the amplitude of the perturbation (for the auxiliary system) grows when considered resonance approaches the separatrices. However, additional analysis allows us to split this perturbation into a Hamiltonian part that does not contribute to capture into resonances and a dissipative part that does not grow near separatrices. One interesting corollary of our analysis is that auxiliary systems describing resonances such that capture can happen converge to a finite number of limit systems, which can be written in terms of Melnikov function [28] describing the splitting of separatrices due to the perturbation. Finally, there is also a small zone near the separatrices where different resonant zones overlap. In this zone the dynamics is very complicated due to Chirikov criterion [29], but, fortunately, this zone is so small that a volume argument yields a good estimate for the time spent in this zone by most solutions.

Future plans. In an ongoing project with Dmitry Dolgopyat, Bassam Fayad, and Jaime Paradela, we study averaging for time intervals longer than $O(\varepsilon^{-1})$. While for time intervals of order ε^{-1} getting estimates on the accuracy of averaging requires only estimating the amplitude of scattering on each resonance and adding those together, longer time intervals require a much more delicate approach that takes into account how scattering on different resonances is related. This was done by D. Dolgopyat [30] in a setting when there is only a finite number of resonances (but each of them is crossed many times, and the number of crossings goes to infinity when $\varepsilon \to 0$). In the setup of [30] the system has an *adiabatic invariant*, i.e., a quantity that is constant along the solutions of averaged system[9]; however, scattering on resonances leads to a jump of the adiabatic invariant of order $\sqrt{\varepsilon}$ each time the system crosses a resonance; those jumps lead to an evolution of the adiabatic observed on time intervals $\gg \varepsilon^{-1}$. Dolgopyat established that different jumps of resonances can be treated as "independent random variables", in particular, their sum satisfies the law of large numbers. This gives a long-term description of the evolution of the adiabatic invariant. The goal of our project is to obtain a similar description for general two-frequency systems. The main novelty of this setting is that, typically, resonances occur at all rational values of ω_2/ω_1 compared to a discrete set in [30]. We also hope that the techniques we develop will contribute to the study of random perturbations of standard-like maps [31, 32]. Standard-like maps naturally appear as Poincaré maps between different resonances, and, as for the random perturbations setting, a different map is applied each time.

3.4 Second order averaging and formulas for crossing phase

While the standard, or first order, averaging method which is described above allows to track the evolution of slow variables (I and z in the notation of Section 3.2), it cannot be used to track the evolution of the fast variable φ. This variable φ is the angle from the pair of action-angle variables, it is also commonly called the *phase*. Tracking its evolution is important in many applied problems: for instance, tracking the phase allows one to determine which of the two potential wells the particle will be captured into for the example in Figure 4a. The standard tool used to this end is the *second order averaging method*. For perturbations of one-frequency systems far from separatrices, it allows to track the slow variables with an error $O(\varepsilon^2)$ and the phase with an error $O(\varepsilon)$. However, the order two averaged system by itself does not predict the phase after a solution crosses a separatrix or (for two or more frequencies) a resonance. In fact, the phase is not even defined on a separatrix. To address this difficulty, a *crossing phase* is introduced; it is a variable containing the information

[8] As they are quite cumbersome, we refer the reader to [20] for a precise statement.

[9] Such systems are very frequent in applications, e.g., systems with the Hamiltonian slowly depending on the time or slow-fast Hamiltonian systems. For these cases, adiabatic invariants are the action variables of the unperturbed system.

on the fast variables at the moment of separatrix crossing. Knowing the crossing phase allows one to track the phase φ after a separatrix crossing. Formulas for crossing phase are known in several different settings when the perturbed system is also Hamiltonian ([24] and references therein). Such formulas are important for the study of probabilistic phenomena associated with separatrix crossing, as the phase determines the domain where the trajectory is captured after a separatrix crossing when capture into multiple domains is possible, as well as the jump of slow variables caused by a separatrix crossing.

In [22] (joint with A. Neishtadt), we estimate the accuracy of the second order averaging method for small perturbations of one-frequency Hamiltonian systems near separatrices and prove a formula for crossing phase. Unlike the previous results, we do this for arbitrary (i.e., possibly non-Hamiltonian) perturbations of one-frequency systems. In [33] (joint with Yuyang Gao and A. Neishtadt) we prove a formula for the phase at *resonant crossing* for slow-fast Hamiltonian systems with one fast angular variable (phase) whose frequency vanishes on some surface in the space of slow variables (a resonant surface). This formula was suggested in [34] on the basis of a heuristic consideration; we prove it rigorously and estimate its accuracy.

Future plans. An ongoing project with Anatoly Neishtadt, Stas Minkov, and Ivan Shilin is to obtain a formula for the phases on a resonance for perturbations of two-frequency systems. This question is open even for the case of just one resonance, and a necessary first step in this direction is improving the accuracy of the existing formulas ([33] and references therein) for the phase on a resonance in the one-frequency case, which is of independent interest and should also allow us to improve the accuracy of the existing formulas for the change of slow variables due to crossing a resonance.

Another open problem is to study consecutive separatrix crossings when the perturbed system is not Hamiltonian. For the Hamiltonian case there are formulas (e.g., [35]) for the jump of slow variables due to separatrix crossing, while it is still an open question to find an analogue of these formulas when the perturbation is non-Hamiltonian. Together with our formula for the phase change, this will enable the study of consecutive separatrix crossings for the general case, as done for the Hamiltonian case in [36, 37].

4 Discrete time dynamics: attractors, skew products, and iterated function systems

4.1 Transitive Anosov diffeomorphism with an invariant horseshoe attracting a set of full Lebesgue measure

Consider a dynamical system $F : X \mapsto X$ and an invariant probability measure μ. The *basin* of μ is the set of all points $x \in X$ such that the sequence of measures $\delta_x^n = \frac{1}{n}(\delta_x + \cdots + \delta_{F^{n-1}(x)})$ converges to μ in the weak-* topology. A measure is called *physical* if its basin has positive Lebesgue measure. By classical results of Sinai, Ruelle and Bowen, any C^2-smooth transitive Anosov diffeomorphism has a unique physical measure supported on the whole phase space and having the basin with full Lebesgue measure. So, the attractor "from the point of view of Lebesgue measure"[10] is the whole phase space as well. When the diffeomorphism is only C^1-smooth, there is a classical example of Bowen [38] of an Anosov diffeomorphism on the 2-torus that has an invariant horseshoe with positive Lebesgue measure. This example implies that the description of physical measures above does not hold in the C^1 case, however, one can still ask if the attractor "from the point of view of Lebesgue measure" is necessarily the whole phase space. Motivated by this question, we construct a counterexample in a joint paper [39] with C. Bonatti, S. Minkov, and I. Shilin. Our example is a *transitive C^1-smooth Anosov diffeomorphism on the 2-torus* such that

- It has an invariant horseshoe and a physical measure supported on this horseshoe. The horseshoe has zero Lebesgue measure.
- The basin of attraction of this physical measure has full Lebesgue measure.
- For almost any point $x \in \mathbb{T}^2$ with respect to the Lebesgue measure, the forward orbit of x is attracted to the horseshoe (i.e., outside of any neighborhood of the horseshoe there are just finitely many points of the orbit).

Our example can be seen as complementary to the example by C. Robinson and L.S. Young [40] of an Anosov diffeomorphism with nonabsolutely continuous stable foliation, also for a C^1-smooth Anosov diffeomorphism on the 2-torus. While the example of Robinson and Young has a horseshoe with positive Lebesgue measure such that its stable foliation has zero measure outside some small neighborhood of the horseshoe (and hence is not absolutely continuous), in our example the horseshoe has zero Lebesgue measure, but its stable foliation has full measure.

[10] One possible formal definition of such attractor is the *Milnor attractor* discussed below in Section 4.3.

Our construction is based on the example by Bowen [38], which can be easily modified to get a horseshoe that is a product of a positive measure Cantor set in the unstable direction and a zero measure Cantor set in the stable direction. It also follows from this construction that the local stable foliation of the horseshoe has positive Lebesgue measure and the horseshoe supports a physical measure attracting a full measure subset of the local stable foliation. The hard part of our paper is to make sure the global stable foliation of the horseshoe has *full and not just positive Lebesgue measure*; control of distortion technique cannot be used to this end as the map is only C^1-smooth. At this point our construction becomes implicit and rather technical: we construct a class of Anosov diffeomorphism having an invariant horseshoe as described, and prove that on a residual subset of this class the stable foliation of the horseshoe has full Lebesgue measure.

4.2 Iterated function systems on the circle

An *iterated function system* (IFS) is a tuple of maps $f_1, \dots, f_k$ on a manifold M; $f_i : M \mapsto M$. In a joint project [41] with Victor Kleptsyn and Yury Kudryashov we study topological properties of IFS's on the circle such that the maps f_i are orientation preserving diffeomorphisms or homeomorphisms.

We prove a general alternative holding on an open and dense subset of C^r-smooth IFS's for any $r \geq 1$:

- either the IFS has an *absorbing domain*, i.e., there exists a finite union of disjoint circle arcs U such that $f_i(U) \subset U$ for $i = 1, \dots k$ and U is not the whole circle or the empty set
- or the IFS is *minimal*, i.e., for any $x \in S^1$ its orbit under the semigroup generated by $f_1, \dots, f_k$ is dense. Moreover, it is also *robustly minimal* in the sense that its small (in C^1) perturbations are also minimal.

We also study conditions required for global synchronization for the minimal case and present several examples illustrating possible degenerate dynamics.

Both minimal IFS's and IFS's with absorbing domain are well studied (let us mention [42] for the minimal case and [43] where IFS's on the segment are studied; the absorming domain case is similar to the segment case). Our alternative shows that, generically, they exhaust all possibilities. For the case of two generators f_1, f_2 that are close to rotations, explicit conditions for minimality were given by Pablo Barrientos and Artem Raibekas [44] in terms of the periodic orbits of the generators. Also, very recently ([45], in preparation) Spencer Durham and Todd Fisher gave similar conditions for minimality for an arbitrary number of generators. Our result complements this picture: while it only works for generic IFS's (i.e. for an open and dense subset), it works for any number of generators that can be arbitrary far from rotations, and the condition for minimality is very easy and natural (no absorbing domain).

While the alternative is purely topological, its proof uses **random dynamics** generated by the IFS: we equip the maps f_i with positive probabilities (the particular choice of the probabilities is not important) and consider random iterations. The tools we use are the properties of the stationary measures for this random dynamical system (they were described by Dominique Malicet [46] for a very general situation when f_i are just homeomorphisms) as well as the behavior of rotation number (for random iterations) in monotone one-parametric families of IFS's.

Future plans in this area include obtaining a similar alternative (transitivity or an absorbing domain) for partially hyperbolic skew products, a setting closely related with IFS's as discussed below in Section 4.3. It would be important for many applications, e.g. for time-periodic perturbations of chaotic systems.

4.3 Milnor attractors of skew products

Milnor's definition of attractor. For a continuous map $f : M \mapsto M$ on a Riemannian manifold M, the *Milnor attractor* of f is a closed subset of M that attracts almost any point with respect to Lebesgue measure, and contains no smaller subset that does so. This definition due to Milnor [47] (he used the term "likely limit set") looks very natural, but it is quite hard to work with as it mixes topological (it must be closed) and measure-theoretic properties. One particular difficulty is that the Milnor attractor is not necessarily preserved after a conjugacy by a homeomorphism: the example from Section 4.1 has the horseshoe as the Milnor attractor, but is conjugate to a linear Anosov diffeomorphism, which has the whole torus as the attractor.

Step skew products. A *step skew product* is a single map that captures the dynamics of an iterated function system $f_1, \dots, f_k$ on some manifold M. It is a skew product over the left shift σ on the space $\{1, \dots, k\}^{\mathbb{Z}}$ of two-sided sequences such that the fiber map is one of the maps $f_1, \dots, f_k$ depending on the zeroth element of the sequence in the base:

$$(\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x)), \qquad \omega \in \{1, \dots, k\}^{\mathbb{Z}},\ x \in M.$$

Step skew products are often used as a model of partially hyperbolic dynamics, and robust properties found in the class of step skew products can often be recreated for smooth partially hyperbolic skew products and their small perturbations, yielding a property locally generic in the space of all diffeomorphisms (this is done, e.g., in [48]). Milnor's definition of attractor can be naturally generalized to step skew products: one equips the generators f_i with some positive probabilities and takes the product of Bernoulli measure in the base and Lebesgue measure in the fiber as the reference measure.

Attractors of step and partially hyperbolic skew products. Questions about Milnor attractor of skew products with one-dimensional fiber were actively advertised by Yulij Ilyashenko when I was his PhD student. They are motivated by two very unusual properties of Milnor attractor that happen locally generically in the class of *boundary-preserving* skew products with the fiber a segment. The first one is *Lyapunov unstable Milnor attractor*, this happens for I. Kan's intermingled basins example [49]. Another one is *thick Milnor attractor*, i.e. Milnor attractor with a positive, but not full Lebesgue measure (Yu. Ilyashenko, [48]). Let us emphasize that while it is easy to construct a single example of a map having any of these properties, both examples above are locally generic among boundary preserving skew products.

I showed that both examples are not generic among circle skew products; it follows that they are also not generic for the interval fiber if this interval is mapped strictly inside itself by the fiber maps. In [50] Lyapunov stability of Milnor attractor is proved for generic partially hyperbolic skew products with the fiber a circle and a transitive Anosov diffeomorphism in the base, together with an alternative that the attractor has either full or zero measure. In [51] (joint with Ivan Shilin) the same is done for generic step skew products with circle fiber. The main ingredients of the proofs are a semicontinuity argument and the fact that the Milnor attractor is saturated by the unstable leaves.

Finally, I would like to mention a natural question in this field remaining wide open: is it true that generically the Milnor attractor of a skew product with circle fiber (either step or partially hyperbolic) is not only Lyapunov stable, but asymptotically stable as well (i.e., is it true that the Milnor attractor actually attracts all points in its small neighborhood).

Preprints

M. Levi, A. Okunev, *Thick Arnold tongues,* arXiv:2411.00175

A. Neishtadt, A. Okunev, *Averaging and passage through resonances in two-frequency systems near separatrices,* arXiv:2108.08540

V. Kleptsyn, Yu. Kudryashov, A. Okunev, *Classification of generic semigroup actions of circle diffeomorphisms,* arXiv:1804.00951

Publications

Y. Gao, A. Neishtadt, A. Okunev, *On phase at a resonance in slow-fast Hamiltonian systems,* **Regular and Chaotic Dynamics**, 28.4 (2023): 585-612. arXiv:2212.13293

S. Minkov, A. Okunev, I. Shilin, *Attractors with Non-Invariant Interior,* **Moscow Mathematical Journal**, 23.4 (2023): 559-570. arXiv:2305.08582

A. Neishtadt, A. Okunev, *Phase change and order 2 averaging for one-frequency systems with separatrix crossing,* **Nonlinearity**, 35.8 (2022): 4469. arXiv:2003.05828

C. Bonatti, S. Minkov, A. Okunev, I. Shilin, *A C^1 Anosov diffeomorphism with a horseshoe that attracts almost any point,* **Discrete & Continuous Dynamical Systems**, 40.1 (2020): 441. arXiv:1802.03977

A. Okunev, *Milnor Attractors of Skew Products with the Fiber a Circle,* **Journal of Dynamical and Control Systems**, 23:2 (2017): 421-433. arXiv:1508.02132

A. Okunev and I. Shilin, *On the attractors of step skew products over the Bernoulli shift,* **Proceedings of the Steklov Institute of Mathematics**, 297 (2017): 260-280. arXiv:1703.01763

S. Minkov, A. Okunev, *Omega-limit sets of generic points of partially hyperbolic diffeomorphisms,* **Functional Analysis and Its Applications**, 50.1 (2016): 48-53.

V. Kleptsyn, A. Okunev, I. Schurov, D. Zubov, M. I. Katsnelson, *Chiral tunneling through generic one-dimensional potential barriers in bilayer graphene*, **Physical Review B**, 92:16 (2016), 165407 arXiv:1507.07638

References

[1] Mark Levi and Alexey Okunev. "Thick Arnold tongues". *arXiv preprint arXiv:2411.00175* (2024).

[2] M.R. Maxey and S. Corrsin. "Gravitational settling of aerosol particles in randomly oriented cellular flow fields". *Journal of the atmospheric sciences* 43.11 (1986), pp. 1112–1134.

[3] H Stommel. "Trajectories of small bodies sinking slowly through convection cells". *Journal of Marine Research* 8 (1949), pp. 24–29.

[4] Martin R Maxey and James J Riley. "Equation of motion for a small rigid sphere in a nonuniform flow". *The Physics of Fluids* 26.4 (1983), pp. 883–889.

[5] Antoine Renaud and Jacques Vanneste. "Dispersion of inertial particles in cellular flows in the small-Stokes, large-Péclet regime". *Journal of Fluid Mechanics* 903 (2020), A2.

[6] H Shin and MR Maxey. "Chaotic motion of nonspherical particles settling in a cellular flow field". *Physical Review E* 56.5 (1997), p. 5431.

[7] MF Piva and S Gabbanelli. "A single dumbbell falling under gravity in a cellular flow field". *Journal of Physics A: Mathematical and General* 36.15 (2003), p. 4291.

[8] Claudia Venditti, Massimiliano Giona, and Alessandra Adrover. "Invariant manifold approach for quantifying the dynamics of highly inertial particles in steady and time-periodic incompressible flows". *Chaos: An Interdisciplinary Journal of Nonlinear Science* 32.2 (2022).

[9] Sergei Petrovich Novikov. "The Hamiltonian formalism and a many-valued analogue of Morse theory". *Russian mathematical surveys* 37.5 (1982), p. 1.

[10] A. Zorich V.I. Arnold M. Kontsevich. "Pseudoperiodic topology". American Mathematical Soc., 1999. Chap. How do the leaves of a closed 1-form wind around a surface?

[11] IA Dynnikov, A Ya Maltsev, and SP Novikov. "Chaotic Trajectories on Fermi Surfaces and Nontrivial Modes of Behavior of Magnetic Conductivity". *Journal of Experimental and Theoretical Physics* 135.2 (2022), pp. 240–254.

[12] Vladimir I Arnold. "Small denominators. I. Mapping the circle onto itself". *Izv. Akad. Nauk SSSR Ser. Mat* 25.1 (1961), pp. 21–86.

[13] J Rubin, CKRT Jones, and M Maxey. "Settling and asymptotic motion of aerosol particles in a cellular flow field". *Journal of Nonlinear Science* 5 (1995), pp. 337–358.

[14] N. Fenichel. "Persistence and Smoothness of Invariant Manifolds for Flows". *Indiana University Mathematics Journal* 21.3 (1971), pp. 193–226.

[15] Colin Boyd. "On the structure of the family of Cherry fields on the torus". *Ergodic Theory and Dynamical Systems* 5.1 (1985), pp. 27–46.

[16] JJP Veerman. "Irrational rotation numbers". *Nonlinearity* 2.3 (1989), p. 419.

[17] Jack Wisdom. "A perturbative treatment of motion near the 3/1 commensurability". *Icarus* 63.2 (1985), pp. 272–289.

[18] F Jay Bourland, Richard Haberman, and William L Kath. "Averaging methods for the phase shift of arbitrarily perturbed strongly nonlinear oscillators with an application to capture". *SIAM Journal on Applied Mathematics* 51.4 (1991), pp. 1150–1167.

[19] Anatoly Neishtadt. "Averaging method for systems with separatrix crossing". *Nonlinearity* 30.7 (2017), p. 2871.

[20] Anatoly Neishtadt and Alexey Okunev. "Averaging and passage through resonances in two-frequency systems near separatrices". *arXiv preprint arXiv:2108.08540* (2021).

[21] Anatoly Neishtadt. "Passage through resonances in a two-frequency problem". *Akademiia Nauk SSSR Doklady*. Vol. 221. 1975, pp. 301–304.

[22] Anatoly Neishtadt and Alexey Okunev. "Phase change and order 2 averaging for one-frequency systems with separatrix crossing". *Nonlinearity* 35.8 (2022), p. 4469.

[23] John R Cary and Rex T Skodje. "Phase change between separatrix crossings". *Physica D: Nonlinear Phenomena* 36.3 (1989), pp. 287–316.

[24] Anatoly Neishtadt and Alexei Vasiliev. "Phase change between separatrix crossings in slow–fast Hamiltonian systems". *Nonlinearity* 18.3 (2005), p. 1393.

[25] Anatoly Neishtadt. "Averaging method for systems with separatrix crossing". *Nonlinearity* 30.7 (2017), p. 2871.

[26] Anatoly Neishtadt. “On the change in the adiabatic invariant on crossing a separatrix in systems with two degrees of freedom”. *Prikl. Mat. Mekh.* 51.5 (1987), p. 750. [Engl. transl.: J. Appl. Math. Mech. 51, No. 5, 586-592 (1987)].

[27] Anatoly Neishtadt. “Averaging, passage through resonances, and capture into resonance in two-frequency systems”. *Russian Mathematical Surveys* 69.5 (2014), p. 771.

[28] V.K. Melnikov. “On the stability of a center for time-periodic perturbations”. *Trudy moskovskogo matematicheskogo obshchestva* 12 (1963), pp. 3–52.

[29] Boris V Chirikov. “Resonance processes in magnetic traps”. *The Soviet Journal of Atomic Energy* 6.6 (1960), pp. 464–470.

[30] Dmitry Dolgopyat. “Repulsion from resonances”. *Mémoires de la Société Mathématique de France* 1 (2012).

[31] Alex Blumenthal, Jinxin Xue, and Lai-Sang Young. “Lyapunov exponents for random perturbations of some area-preserving maps including the standard map”. *Annals of Mathematics* 185.1 (2017), pp. 285–310.

[32] Alex Blumenthal, Jacopo De Simoi, and Ke Zhang. “Diffusion limit for a slow-fast standard map”. *Communications in Mathematical Physics* 374.1 (2020), pp. 187–210.

[33] Yuyang Gao, Anatoly Neishtadt, and Alexey Okunev. “On phase at a resonance in slow-fast Hamiltonian systems”. *Regular and Chaotic Dynamics* 28.4 (2023), pp. 585–612.

[34] AV Artemyev, AI Neishtadt, and AA Vasiliev. “Mapping for nonlinear electron interaction with whistler-mode waves”. *Physics of Plasmas* 27.4 (2020).

[35] John Cary, Dominique Escande, and Jonathan Tennyson. “Adiabatic-invariant change due to separatrix crossing”. *Physical Review A* 34.5 (1986), p. 4256.

[36] Analoly Neishtadt, Vladislav Sidorenko, and Dmitry Treschev. “Stable periodic motions in the problem on passage through a separatrix”. *Chaos* 7.1 (1997), p. 2.

[37] A.A. Vasiliev, A.I. Neishtadt, C. Simó, and D.V. Treschev. “Stability islands in domains of separatrix crossings in slow-fast Hamiltonian systems”. *Proceedings of the Steklov Institute of Mathematics* 259.1 (2007), p. 236.

[38] Rufus Bowen. “A horseshoe with positive measure”. *Inventiones mathematicae* 29.3 (1975), pp. 203–204.

[39] Christian Bonatti, Stanislav Minkov, Alexey Okunev, and Ivan Shilin. “Anosov diffeomorphism with a horseshoe that attracts almost any point”. *Discrete & Continuous Dynamical Systems-A* 40.1 (2020), p. 441.

[40] Clark Robinson and Lai-Sang Young. “Nonabsolutely continuous foliations for an Anosov diffeomorphism”. *Inventiones mathematicae* 61.2 (1980), pp. 159–176.

[41] Victor Kleptsyn, Yury Kudryashov, and Alexey Okunev. “Classification of generic semigroup actions of circle diffeomorphisms”. *arXiv preprint arXiv:1804.00951* (2018).

[42] VA Antonov. “Modeling cyclic evolution processes: Synchronization by means of a random signal”. *Leningradskii Universitet Vestnik Matematika Mekhanika Astronomiia* (1984), pp. 67–76.

[43] V Kleptsyn and D Volk. “Physical Measures for Nonlinear Random Walks on Interval”. *Moscow Mathematical Journal* 14.2 (2014), pp. 339–365.

[44] Pablo G Barrientos and Artem Raibekas. “Dynamics of iterated function systems on the circle close to rotations”. *Ergodic Theory and Dynamical Systems* 35.5 (2015), pp. 1345–1368.

[45] Spencer Durham and Todd Fisher. “Minimality for iterated function systems of circle diffeomorphisms”. *in preparation* (2024).

[46] Dominique Malicet. “Random walks on Homeo(S^1)”. *Communications in Mathematical Physics* 356 (2017), pp. 1083–1116.

[47] John Milnor. “On the concept of attractor”. *The theory of chaotic attractors*. Springer, 1985, pp. 243–264.

[48] Yulij Ilyashenko. “Thick attractors of boundary preserving diffeomorphisms”. *Indagationes Mathematicae* 22.3-4 (2011), pp. 257–314.

[49] Ittai Kan. “Open sets of diffeomorphisms having two attractors, each with an everywhere dense basin”. *Bulletin of the American Mathematical Society* 31.1 (1994), pp. 68–74.

[50] Alexey Okunev. “Milnor attractors of skew products with the fiber a circle”. *Journal of Dynamical and Control Systems* 23.2 (2017), pp. 421–433.

[51] Alexey Okunev and Ivan Shilin. “On the attractors of step skew products over the Bernoulli shift”. *Proceedings of the Steklov Institute of Mathematics* 297.1 (2017), pp. 235–253.