

Research statement

Alexey Okunev
aokunev@umd.edu

1 Overview

My research interest is in dynamical systems, with an emphasis on physically motivated problems. Most of my recent work is in perturbations of Hamiltonian systems, where a small perturbation of a completely integrable system can produce chaotic properties such as sensitivity to initial data and random phenomena. I also have a broad background and worked on problems involving iterated function systems, attractors, and partial and nonuniform hyperbolicity. For example, we constructed a C^1 -smooth Anosov diffeomorphism with a physical measure supported on a set with zero Lebesgue measure but attracting a set of full Lebesgue measure. In the future, I plan to work towards a rigorous understanding of diffusion on resonances for perturbations of two-frequency systems, a phenomenon widely accepted in physics but largely open mathematically.

2 Weak mixing for area preserving surface flows

Smooth area-preserving flows on a compact, connected, orientable surface are among the simplest and most natural classes of dynamical systems. S.P. Novikov [1] initiated a deep study of such flows, motivated by connections with a problem arising in solid-state physics, namely, the motion of an electron in a metal under the action of a magnetic field; see a recent survey [2] for more on this topic. Topologically, an area-preserving flow on a surface decomposes [3, 4] into a union of domains filled by periodic trajectories and domains where all but finitely many trajectories are dense. The latter are called *(quasi)-minimal components*, and we focus on them from now on. For each minimal component, one can select a transversal such that the first return map to it is an interval exchange transformation (IET); it is a bijection of the interval into itself with finitely many discontinuities which is locally a translation. Together with the return time to this transversal, the IET fully describes ergodic properties of the flow restricted to the minimal component. Note that generally a minimal component can break down into several ergodic components, but for typical IETs there is just one such component [5, 6], then a minimal component is uniquely ergodic.

However, mixing properties of area-preserving surface flows turned out to be quite delicate. A flow ϕ_t preserving a probability measure μ is *mixing* if for each pair A, B of measurable sets $\lim_{t \rightarrow \infty} |\mu(\phi_t(A) \cap B) - \mu(A)\mu(B)| = 0$. Intuitively, this means that any measurable set equidistributes on the whole phase space after a long time. If one only has the convergence above on average: $\lim_{T \rightarrow \infty} (1/T) \int_0^T |\mu(\phi_t(A) \cap B) - \mu(A)\mu(B)| dt = 0$, such a flow is called *weakly mixing*. Intuitively, sets still equidistribute for most large times, but not necessarily for all of them. Flows without fixed points only exist on the torus and are never mixing. However, they may be weakly mixing depending on arithmetic properties of the frequency (i.e., asymptotic slope of the trajectories): they are never weakly mixing if the frequency is Diophantine [7], but may be weakly mixing if it is Liouvillean [8]. The first example of a mixing area-preserving surface flow is due to Kochergin [9]; mixing arises due to the slowing down of trajectories near fixed points of the flow and the corresponding infinite singularities of the first return time to a transversal. Much more recently, Ulcigrai studied mixing for typical flows with nondegenerate fixed points, where typical means for almost every choice of interval lengths of the corresponding IET. There are two cases based on the combinatorics of separatrix connections; in the so-called "symmetric" case a typical flow is not mixing [10], and in the "asymmetric" case a typical flow is mixing [11]. However, even in the symmetric case a typical flow is weakly mixing [12]. Chaika and Wright [13] constructed a mixing flow in the symmetric case, complementing [10]. The case of degenerate saddles first appeared in Kochergin's mixing example [9]; Frączek and Kim [14] studied ergodic properties of flows with degenerate fixed points, allowing for saddles with any multiplicity, but requiring them to be in a special class (e.g., not all polynomial saddles are allowed).

In a joint work with A. Kanigowski and R. Zelada [15], we show that **a single fixed point already implies weak mixing**. Namely, under the assumption that the flow has at least one but finitely many fixed points and is analytic near each saddle fixed point, we prove that it is weakly mixing on each minimal component¹. This is in sharp contrast with flows without fixed points, where weak mixing depends on the arithmetic properties of the frequency. It is also, as far as we know, the first result in the field covering arbitrary analytic fixed points (e.g., a "monkey" saddle with six separatrices meeting at the case when some separatrices are tangent to each other). The analyticity assumption is used to describe the time it takes for a trajectory to pass through a small neighborhood of a degenerate saddle fixed point using resolution of singularities. We also separately cover the case when all fixed points are C^∞ only, but nondegenerate, upgrading Ulcigrai's result [12] from typical to all flows.

3 Thick Arnold tongues

In a joint project with Mark Levi [16], we study a physically motivated dynamical system exhibiting nontrivial and perhaps unexpected properties. The mathematical explanation involves perturbation theory and circle maps dynamics, and this system also has connections with number theory and topology. The example we study originates from particle transport in fluid flows: it is a simplified mathematical model of a small inertial particle drifting in a fluid and pulled by gravity proposed by Maxey and Corrsin in 1986, [17]. The fluid motion in this example is given by the *cellular flow*, a periodic Hamiltonian flow on the plane along the level curves of $H = \cos x \cos y$ (Figure 1a).

¹Each minimal component decomposes into finitely many ergodic pieces; we show weak mixing when restricted to those.

We are interested in the case when the cells are not aligned with gravity. As the vertical direction is already booked by cell orientation, the direction of the gravity becomes a parameter given by a vector $(b, a) \in \mathbb{R}^2$. We call this vector *forcing*, and it can be interpreted as terminal fall velocity of the particle in still fluid.

When the particle is so small that the effect of its inertia is negligible (*zero inertia case*), its movement is given by the following Hamiltonian vector field:

$$\mathbf{v}(x, y) = \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x} \right), \quad H(x, y) = \cos x \cos y - ax + by. \quad (1)$$

This model (1) was introduced in the context of particle transport as early as 1949 by Stommel [18]. A particle with small but nonzero inertia has a velocity of its own; however, the fluid drag force proportional to the velocity mismatch between the particle and surrounding fluid makes its motion close to the motion of a zero inertia particle.² Let $\mathbf{x} = (x, y)$ and $\dot{\mathbf{x}} = (\dot{x}, \dot{y})$ denote the particle's position and velocity and let $\varepsilon > 0$ be a small parameter describing the inertia strength. Then particle's movement can be modeled by a second order ordinary differential equation:

$$\ddot{\mathbf{x}} = -\frac{1}{\varepsilon}(\dot{\mathbf{x}} - \mathbf{v}(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^2. \quad (2)$$

Such a model is a simplified case of the Maxey-Riley model [19] and is often used in the literature on particle transport; cellular flow is also a very common choice of the fluid flow. While a spherical particle moving in a cellular flow without the gravity would simply approach the cell boundaries, many variations of this setup exhibit nontrivial dynamics. To name just a few references (more can be found therein), there are studies of particles in a cellular flow that are influenced by random thermal noise [20], are asymmetric [21] or compound [22], as well as studies of particles in time-dependent cellular flow [23].

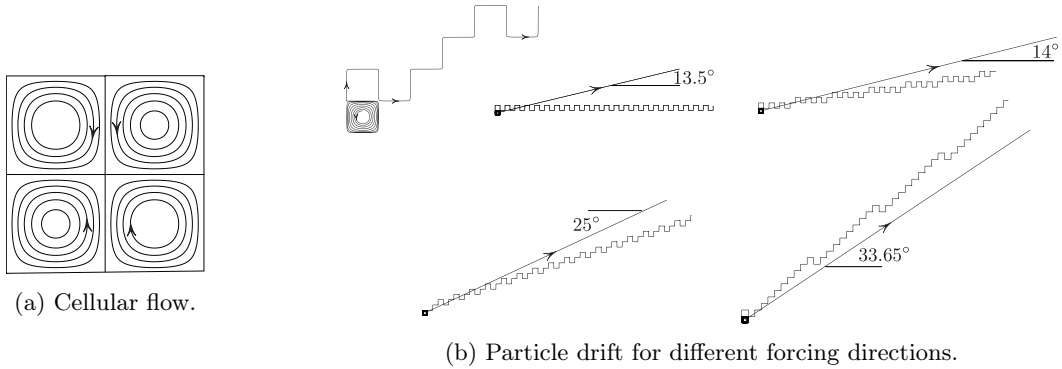


Figure 1: Cellular flow and particle drift according to (2).

To describe the particle motion given by (2) and (1), consider the trajectories of the Hamiltonian vector field $\mathbf{v}$ first; those are simply level curves of $H(x, y)$. Assuming that the forcing (b, a) is fairly small, most of the trajectories are bounded, but some "channels" formed by unbounded trajectories appear. All unbounded trajectories are at a finite distance from a straight line with the slope a/b , as $H = -ax + by + O(1)$ is constant on the trajectories. As a side note, the combinatorics of these unbounded trajectories is quite intriguing and is connected to symbolic dynamics as well as to number-theoretical properties of b/a . Our interest in this problem was motivated by unusual properties of *inertial* particle motion, as modeled by (2) with $\mathbf{v}$ given by (1), that we first observed in numerical experiments. While inertial particle trajectories also have an asymptotic slope that does not depend on the initial data, the direction of the particle drift does not coincide with the forcing direction (Figure 1b). Moreover, as Figure 1b (top) illustrates, a small 0.5° change in the direction of the forcing causes a drastic change of the direction of the trajectory.

We **investigate how the direction of particle drift depends on the slope a/b of the external forcing** and prove that when a and b are positive but fairly small³ for any small enough $\varepsilon > 0$

- Particle drift slope ρ exists and does not depend on the initial data $(\mathbf{x}(0), \dot{\mathbf{x}}(0))$, given that $(\mathbf{x}(0), \dot{\mathbf{x}}(0))$ is outside of a very small subset of $\mathbb{R}^4$ given by a countable union of codimension one manifolds.
- For fixed b and ε , the dependence of particle drift slope on the forcing slope $\alpha = \frac{a}{b}$ is a Cantor-like function with steps at all rational values of ρ (Figure 2a).
- The function $\rho(\alpha)$ is steep at the endpoints of the steps: if one takes α_0 at the right endpoint of any step, for any $n \in \mathbb{N}$ we have $\lim_{\Delta\alpha \rightarrow 0^+} \frac{(\rho(\alpha_0 + \Delta\alpha) - \rho(\alpha_0))^n}{\Delta\alpha} = \infty$.⁴ This explains the abrupt changes of ρ we observed in numerics.

²A simple computation shows that the total force acting on the particle (the fluid drag plus gravity) is proportional to the difference between the actual particle velocity and the velocity $\mathbf{v}$ a zero-inertia particle would have.

³We assume $a, b \in (\gamma, \delta)$ with δ small enough; then there exists $\varepsilon_1 = \varepsilon_1(\gamma, \delta)$ such that conclusions hold for all positive $\varepsilon < \varepsilon_1$.

⁴Which is like the inverse function to the classic Calculus example $y = e^{-1/x}$. It was a pleasant surprise to find such a function outside a textbook.

- The set of the values of α such that $\rho(\alpha)$ is irrational (equivalently, the complement to the union of the steps) has zero Hausdorff dimension (and hence zero Lebesgue measure as well).

In the above description, the parameter ε in (2) was fixed. A fuller picture can be seen when ε is varied as well: for each rational number $\frac{p}{q}$, one can draw the set (called a *tongue*) of all pairs (ε, α) giving particle drift slope equal to $\frac{p}{q}$ (Figure 2b). The picture we get is similar to the famous Arnold tongues example [24], but, unlike it, the union of all tongues occupies a full Lebesgue measure – hence we use the name "thick Arnold tongues" for this example.

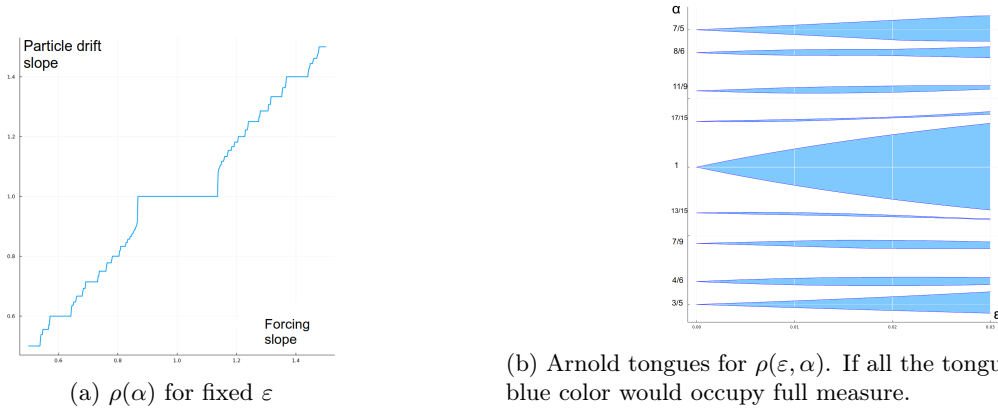


Figure 2: Particle drift slope ρ as a function of forcing slope α and inertia parameter ε .

This description is obtained using tools from perturbation theory and circle maps dynamics. As observed by Rubin, Jones, and Maxey [25] using Fenichel's results [26] on normal hyperbolicity, the system (2) in $\mathbb{R}^4 \ni (\mathbf{x}, \dot{\mathbf{x}})$ can be reduced to a flow on the 2-torus. Taking a first return map further reduces the problem to the study of a family of circle maps possessing a special feature: they have two *flat spots*, that is, two intervals where these maps are constant. The particle drift slope coincides with the rotation number of the circle map up to a linear scaling. Circle maps with flat spots were considered by many authors and appear naturally as first return maps for Cherry flows on torus and as truncations of non-invertible circle endomorphisms. We build on the works of Boyd [27] and Veerman [28] to understand how the rotation number depends on the parameter α in the case at hand. This involves a careful study of the properties of the circle map family originating from the model of particle transport and generalizing Veerman's theorem [28] claiming zero Hausdorff dimension of the set of parameters with irrational rotation number for families of circle maps with one flat spot to the case of an arbitrary number of flat spots.

4 Averaging and separatrix crossings

Separatrix crossings for small perturbations of integrable systems are important in many applications such as celestial mechanics and dynamics of charged particles. My favorite example is the work [29] by Jack Wisdom describing how consecutive separatrix crossings change the orbits of asteroids in a resonance with Jupiter resulting in the formation of the Kirkwood gaps within the main asteroid belt.⁵ To introduce separatrix crossings for readers with interests far from perturbation theory, let us examine the following example. Consider a particle with position and velocity $x, \dot{x} \in \mathbb{R}$ in a double-well potential $V(x)$, Figure 3a. Then let us add to this *unperturbed system* a small dissipative force εf acting on it, called the *perturbation*:

$$\ddot{x} = -\frac{\partial V}{\partial x} + \varepsilon f(x, \dot{x}). \quad (3)$$

The unperturbed system is a completely integrable Hamiltonian system, and the energy $E = \frac{\dot{x}^2}{2} + V(x)$ is a first integral. To study the change of energy due to the perturbation, one can average the rate of change of E over periodic solutions of the unperturbed system and get a one-dimensional ODE. This system is called the *averaged system*, Figure 3b. If the energy for the averaged system decreases (e.g., f includes friction), then the particle will eventually be trapped into one of the two wells. This is called separatrix crossing, as the solution of (3) goes through a separatrix of the unperturbed system, the figure-eight shape in Figure 3a. Note that the particle can be captured into each of the two wells⁶, so one has to consider two solutions of the averaged system. Then one of them will be close to the actual evolution of the energy, with the error $O(\varepsilon \ln \varepsilon)$, [30, 31]. One also has to remove a small measure of initial data that "cannot decide" between the two wells and gets trapped near the point C between them.

⁵This mechanism is also considered to be a possible source of Earth-crossing asteroids.

⁶And order one averaging method cannot determine which well it will be captured in.

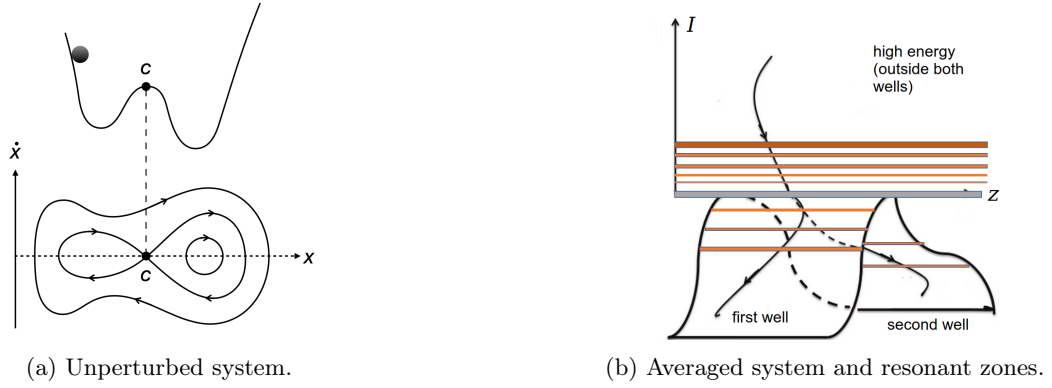


Figure 3: Example of separatrix crossing.

Suppose now that the perturbing force periodically depends on the time: $\ddot{x} = -\frac{\partial V}{\partial x} + \varepsilon f(x, \dot{x}, t)$ with $f(x, \dot{x}, t) = f(x, \dot{x}, t + 1)$. Then, in order to obtain the averaged system, one averages with respect to both φ and t , where $\varphi \in [0, 2\pi]$ is the angle from the pair of action-angle variables. These variables parametrize periodic trajectories of unperturbed system in such a way that φ is constant along each trajectory. The unperturbed dynamics of (φ, t) for fixed E is a linear torus flow, and typically (for irrational winding) the time average will coincide with the spatial average. However, this is not the case if E corresponds to a rational winding of the torus, and this leads to a larger error of averaging. Such values of E are called *resonant*, and when E changes, it crosses resonant values, Figure 3b.⁷ For most initial data, crossing a resonance causes a $O(\sqrt{\varepsilon})$ deviation from the averaged system, but a small measure of initial data can be captured into the resonance – that is, $E(t)$ gets trapped near the resonant value for a time of order $1/\varepsilon$, leading to order 1 deviation from the averaged system.

In [32] (joint with A. Neishtadt, submitted), we rigorously **estimate the accuracy of averaging method for small perturbations of completely integrable two-frequency Hamiltonian systems near separatrices**. One can keep in mind the example above, where the two angular variables are φ and t . However, our setting is perturbations of general two-frequency systems, and we also allow the Hamiltonian to depend on parameters slowly changing due to the perturbation. We show that for any initial data outside a small measure set corresponding to solutions close to capture into a resonance, the corresponding solution of the perturbed system is approximately described by the averaging method with the accuracy $O(\sqrt{\varepsilon})$. This accuracy cannot be improved, as a single passage through resonance causes such error. This result extends Neishtadt's Theorem (1975, [33]) on two-frequency systems far from separatrices to the general case with separatrix crossings and provides rigorous justification for the use of averaging method in this setting, which is frequently encountered in applications. This was a challenging problem with many difficulties to overcome. To name one, far from separatrices there are only finitely many resonances such that capture is possible, while in our setting there can be an infinite number of resonances allowing for capture, and these resonances accumulate on the separatrices.

A side result of the same project with A. Neishtadt is an estimate [34] on the accuracy of order two averaging method for non-Hamiltonian perturbations of one-frequency systems. For the example above, this is the case when f does not depend on t . Order 2 averaging allows one to track the evolution of the angle φ after separatrix crossing, which is impossible using the standard (order one) averaging method. In particular, for our example order 2 averaging describes in which well the particle will be captured. Such formulas for the angle φ after a separatrix crossing were obtained before for various settings when the perturbed system is also Hamiltonian [35, 36].

Future plans. In an ongoing project with Dmitry Dolgopyat, Bassam Fayad, and Jaime Paradela, we study averaging for time intervals longer than $O(\varepsilon^{-1})$. While for time intervals of order ε^{-1} getting estimates on the accuracy of averaging requires only estimating the amplitude of scattering on each resonance and adding those together, longer time intervals require a much more delicate approach that takes into account how scattering on different resonances is related. This was done by D. Dolgopyat [37] in a setting when there is only a finite number of resonances (but each of them is crossed many times, and the number of crossings goes to infinity when $\varepsilon \rightarrow 0$). In the setup of [37] the system has an *adiabatic invariant*, i.e., a quantity that is constant along the solutions of averaged system⁸; however, scattering on resonances leads to a jump of the adiabatic invariant of order $\sqrt{\varepsilon}$ each time the system crosses a resonance; those jumps lead to an evolution of the adiabatic observed on time intervals $\gg \varepsilon^{-1}$. Dolgopyat established that different jumps of resonances can be treated as "independent random variables", in particular, their sum satisfies the law of large numbers. This gives a long-term description of the evolution of the adiabatic invariant. The goal of our project is to obtain a similar description for general two-frequency systems and rigorously justify **diffusion on resonances** in this setting. One novelty of the setting is that, typically, resonances occur at all rational values of ω_2/ω_1 compared to a discrete set in [37]. We also hope that the techniques we develop will contribute to the study of random perturbations of standard-like maps [38, 39]. Standard-like maps naturally appear as Poincaré maps between different resonances, and, as for the random perturbations setting, a different map is applied each time.

⁷one can leave only $O(|\ln \varepsilon|)$ resonant values, as others have negligibly small effect on the dynamics

⁸Such systems are very frequent in applications, e.g., systems with the Hamiltonian slowly depending on the time or slow-fast Hamiltonian systems. For these cases, adiabatic invariants are the action variables of the unperturbed system.

Another ongoing project (with Anatoly Neishtadt, Stas Minkov, and Ivan Shilin) is to obtain a formula for the phases on a resonance for perturbations of two-frequency integrable systems. This question is open even for the case of just one resonance, and a necessary first step in this direction is improving the accuracy of the existing formulas ([40] and references therein) for the phase on a resonance in the one-frequency case, which is of independent interest and should also allow us to improve the accuracy of the existing formulas for the change of slow variables due to crossing a resonance.

Finally, a related important problem is to study consecutive separatrix crossings for the case when the perturbed system is not Hamiltonian. In addition to a formula for the change of phase [34] after a separatrix crossing, this would require obtaining a formula for the jump of slow variables and analyzing the discrete dynamics given by these formulas. For the Hamiltonian case this is done in [41, 42].

5 Transitive Anosov diffeomorphism with an invariant horseshoe attracting a set of full Lebesgue measure

Consider a dynamical system $F : X \mapsto X$ and an invariant probability measure μ . The *basin* of μ is the set of all points $x \in X$ such that the sequence of measures $\delta_x^n = \frac{1}{n}(\delta_x + \dots + \delta_{F^{n-1}(x)})$ converges to μ in the weak-* topology. A measure is called *physical* if its basin has positive Lebesgue measure. By classical results of Sinai, Ruelle and Bowen, any C^2 -smooth transitive Anosov diffeomorphism has a unique physical measure supported on the whole phase space and having the basin with full Lebesgue measure. So, the attractor "from the point of view of Lebesgue measure"⁹ is the whole phase space as well. When the diffeomorphism is only C^1 -smooth, there is a classical example of Bowen [43] of an Anosov diffeomorphism on the 2-torus that has an invariant horseshoe with positive Lebesgue measure. This example implies that the description of physical measures above does not hold in the C^1 case, however, one can still ask if the attractor "from the point of view of Lebesgue measure" is necessarily the whole phase space. Motivated by this question, we construct a counterexample in a joint paper [44] with C. Bonatti, S. Minkov, and I. Shilin. Our example is a *transitive C^1 -smooth Anosov diffeomorphism on the 2-torus* such that

- It has an invariant horseshoe and a physical measure supported on this horseshoe. The horseshoe has zero Lebesgue measure.
- The basin of attraction of this physical measure has full Lebesgue measure.
- For almost any point $x \in \mathbb{T}^2$ with respect to the Lebesgue measure, the forward orbit of x is attracted to the horseshoe (i.e., outside of any neighborhood of the horseshoe there are just finitely many points of the orbit).

Our example can be seen as complementary to the example by C. Robinson and L.S. Young [45] of an Anosov diffeomorphism with nonabsolutely continuous stable foliation, also for a C^1 -smooth Anosov diffeomorphism on the 2-torus. While the example of Robinson and Young has a horseshoe with positive Lebesgue measure such that its stable foliation has zero measure outside some small neighborhood of the horseshoe (and hence is not absolutely continuous), in our example the horseshoe has zero Lebesgue measure, but its stable foliation has full measure.

Our construction is based on the example by Bowen [43], which can be easily modified to get a horseshoe that is a product of a positive measure Cantor set in the unstable direction and a zero measure Cantor set in the stable direction. It also follows from this construction that the local stable foliation of the horseshoe has positive Lebesgue measure and the horseshoe supports a physical measure attracting a full measure subset of the local stable foliation. The hard part of our paper is to make sure the global stable foliation of the horseshoe has *full and not just positive Lebesgue measure*; control of distortion technique cannot be used to this end as the map is only C^1 -smooth. At this point our construction becomes implicit and rather technical: we construct a class of Anosov diffeomorphism having an invariant horseshoe as described, and prove that on a residual subset of this class the stable foliation of the horseshoe has full Lebesgue measure.

6 Iterated function systems on the circle

An *iterated function system* (IFS) is a tuple of maps $f_1, \dots, f_k$ on a manifold M ; $f_i : M \mapsto M$. In a joint project [46] with Victor Kleptsyn and Yury Kudryashov we study topological properties of IFS's on the circle such that the maps f_i are orientation preserving diffeomorphisms or homeomorphisms.

We prove a general alternative holding on an open and dense subset of C^r -smooth IFS's for any $r \geq 1$:

- either the IFS has an *absorbing domain*, i.e., there exists a finite union of disjoint circle arcs U such that $f_i(U) \subset U$ for $i = 1, \dots, k$ and U is not the whole circle or the empty set
- or the IFS is *minimal*, i.e., for any $x \in S^1$ its orbit under the semigroup generated by $f_1, \dots, f_k$ is dense. Moreover, it is also *robustly minimal* in the sense that its small (in C^1) perturbations are also minimal.

⁹One possible formal definition of such attractor is the *Milnor attractor* discussed below in Section 7.

We also study conditions required for global synchronization for the minimal case and present several examples illustrating possible degenerate dynamics.

Both minimal IFS's and IFS's with absorbing domain are well studied (let us mention [47] for the minimal case and [48] where IFS's on the segment are studied; the absorbing domain case is similar to the segment case). Our alternative shows that, generically, they exhaust all possibilities. For the case of two generators f_1, f_2 that are close to rotations, explicit conditions for minimality were given by Pablo Barrientos and Artem Raibekas [49] in terms of the periodic orbits of the generators. Also, very recently ([50], in preparation) Spencer Durham and Todd Fisher gave similar conditions for minimality for an arbitrary number of generators. Our result complements this picture: while it only works for generic IFS's (i.e. for an open and dense subset), it works for any number of generators that can be arbitrary far from rotations, and the condition for minimality is very easy and natural (no absorbing domain).

While the alternative is purely topological, its proof uses **random dynamics** generated by the IFS: we equip the maps f_i with positive probabilities (the particular choice of the probabilities is not important) and consider random iterations. The tools we use are the properties of the stationary measures for this random dynamical system (they were described by Dominique Malicet [51] for a very general situation when f_i are just homeomorphisms) as well as the behavior of rotation number (for random iterations) in monotone one-parametric families of IFS's.

Future plans in this area include obtaining a similar alternative (transitivity or an absorbing domain) for partially hyperbolic skew products, a setting closely related with IFS's as discussed below in Section 7. It would be important for many applications, e.g. for time-periodic perturbations of chaotic systems.

7 Milnor attractors of skew products

Milnor's definition of attractor. For a continuous map $f : M \mapsto M$ on a Riemannian manifold M , the *Milnor attractor* of f is a closed subset of M that attracts almost any point with respect to Lebesgue measure, and contains no smaller subset that does so. This definition due to Milnor [52] (he used the term "likely limit set") looks very natural, but it is quite hard to work with as it mixes topological (it must be closed) and measure-theoretic properties. One particular difficulty is that the Milnor attractor is not necessarily preserved after a conjugacy by a homeomorphism: the example from Section 5 has the horseshoe as the Milnor attractor, but is conjugate to a linear Anosov diffeomorphism, which has the whole torus as the attractor.

Attractors of partially hyperbolic skew products. Questions about Milnor attractor of skew products with one-dimensional fiber were actively advertised by Yulij Ilyashenko when I was his PhD student. They are motivated by two very unusual properties of Milnor attractor that happen locally generically in the class of *boundary-preserving* skew products with the fiber a segment. The first one is *Lyapunov unstable Milnor attractor*, this happens for I. Kan's intermingled basins example [53]. Another one is *thick Milnor attractor*, i.e. Milnor attractor with a positive, but not full Lebesgue measure (Yu. Ilyashenko, [54]). Let us emphasize that while it is easy to construct a single example of a map having any of these properties, both examples above are locally generic among boundary preserving skew products.

I showed that both examples are not generic among circle skew products; it follows that they are also not generic for the interval fiber if this interval is mapped strictly inside itself by the fiber maps. In [55] Lyapunov stability of Milnor attractor is proved for generic partially hyperbolic skew products with the fiber a circle and a transitive Anosov diffeomorphism in the base, together with an alternative that the attractor has either full or zero measure. The main ingredients of the proof are a semicontinuity argument and the fact that the Milnor attractor is saturated by the unstable leaves.

Preprints

- A. Kanigowski, A. Okunev, R. Zelada, *Weak mixing for area preserving flows on surfaces*, arXiv:2602.15719
- M. Levi, A. Okunev, *Thick Arnold tongues*, arXiv:2411.00175
- V. Kleptsyn, Yu. Kudryashov, A. Okunev, *Classification of generic semigroup actions of circle diffeomorphisms*, arXiv:1804.00951

Publications

- A. Neishtadt, A. Okunev, *Averaging and passage through resonances in two-frequency systems near separatrices*, **Communications in Mathematical Physics**, 406.3 (2025), p.63. arXiv:2108.08540
- Y. Gao, A. Neishtadt, A. Okunev, *On phase at a resonance in slow-fast Hamiltonian systems*, **Regular and Chaotic Dynamics**, 28.4 (2023): 585-612. arXiv:2212.13293
- S. Minkov, A. Okunev, I. Shilin, *Attractors with Non-Invariant Interior*, **Moscow Mathematical Journal**, 23.4 (2023): 559-570. arXiv:2305.08582
- A. Neishtadt, A. Okunev, *Phase change and order 2 averaging for one-frequency systems with separatrix crossing*, **Nonlinearity**, 35.8 (2022): 4469. arXiv:2003.05828
- C. Bonatti, S. Minkov, A. Okunev, I. Shilin, *A C^1 Anosov diffeomorphism with a horseshoe that attracts almost any point*, **Discrete & Continuous Dynamical Systems**, 40.1 (2020): 441. arXiv:1802.03977
- A. Okunev, *Milnor Attractors of Skew Products with the Fiber a Circle*, **Journal of Dynamical and Control Systems**, 23:2 (2017): 421-433. arXiv:1508.02132
- A. Okunev and I. Shilin, *On the attractors of step skew products over the Bernoulli shift*, **Proceedings of the Steklov Institute of Mathematics**, 297 (2017): 260-280. arXiv:1703.01763
- S. Minkov, A. Okunev, *Omega-limit sets of generic points of partially hyperbolic diffeomorphisms*, **Functional Analysis and Its Applications**, 50.1 (2016): 48-53.
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